LINEAR ANALYSIS OF A TROPICAL CYCLONE MODEL WITH INCREASED VERTICAL RESOLUTION

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ABSTRACT

As a precursor to numerical simulation of tropical cyclones with multilevel vertical resolution, a linear model in which the thermodynamic equation is applied at two levels is considered. Eigenvalue solutions for conditional and unconditional heating are compared.

1. INTRODUCTION

Dynamical models designed for the study of tropical cyclones are crucially dependent on a parametric representation of the large-scale heating function which is produced by organized systems of cumulonimbi. Models in which this heating is related to cyclone-scale convergence in a subcloud friction layer and in which the vertically integrated heating function is dependent upon the flux of latent heat through the top of the friction layer have yielded promising results[1] within the framework of models which have been truncated in the vertical to the extent that the thermodynamic equation is applied at only one level. Examples are [1, 7, 8]. The design of models with greater vertical resolution has proven difficult.

As has been noted by others [1, 2, 7], the vertically integrated heating is fairly well approximated by the flux of latent heat upward through the boundary layer. However, rather little is known about the vertical distribution of this heating. The latter is strongly dependent on the cumulus-scale circulations as well as on entrainment. Different theoretical treatments of the interactions of cumuli with their environments (for examples, [5, 6]) predict substantially different vertical variations of heating. With this in mind, a number of investigators have concluded that tropical cyclone models with increased vertical resolution are not yet within reach.

On the other hand, we do have considerable knowledge concerning the structure of the tropical cyclone and its characteristic time variations. If we can determine vertical distributions of heating such that the solutions to the time dependent equations for the cyclone scale show structures and time scales consistent with the observations, we will have provided useful information for the design of cumulus-cyclone interaction models. Syōno and Yamasaki's [9] linear analysis of a system with two thermodynamic levels and unconditional heating is a valuable step in this direction. The primary purpose of this paper is to extend these results to conditional heating and to attempt a delineation of the vertical distribution of heating necessary for the solutions to resemble tropical cyclones.

2. THE LINEAR MODEL

Figure 1 shows the vertical structure of our model. The thermodynamic equation is applied at levels 2 and 4.

The equations of horizontal motion, in balanced form,[2] and the continuity equation are applied at levels 1, 3, 5, and 7. Friction is included only in the surface layer as is typical of studies of this type.

The thermodynamic equation for a hydrostatic atmosphere, linearized on a stagnant base state, takes the form:

$$\frac{\partial^2\phi}{\partial p\partial t}+\sigma\omega=-\frac{R}{pc_p}\dot{Q}.$$

ϕ and ω are, respectively, the geopotential and vertical motion perturbations; σ is the base-state static stability ($\sigma=-(\bar{\alpha}/\bar{\theta})\partial\bar{\theta}/\partial p$, $\bar{\alpha}$=base-state specific volume, $\bar{\theta}$=base-state potential temperature). $\dot{Q}$ is the heating function per unit time and mass. R, c_p, p, and t have their conventional meanings. With the usual finite difference approximations,

$$\frac{\partial\phi_3}{\partial t}-\frac{\partial\phi_1}{\partial t}+\sigma_2\Delta p\ \omega_2=-\frac{R}{c_p}\dot{Q}_2. \tag{1}$$

[1] The term, "conditional instability of the second kind," has been suggested [1] for growth which results from energy imparted by this mechanism. Hereafter, this will be abbreviated to "CISK."

[2] To the authors' knowledge, Syōno and Yamasaki [9] are the only investigators who have attempted a linear analysis of CISK within the framework of the primitive equations. The mathematical difficulties involved in solving the primitive equations with conditional heating appear to be formidable.

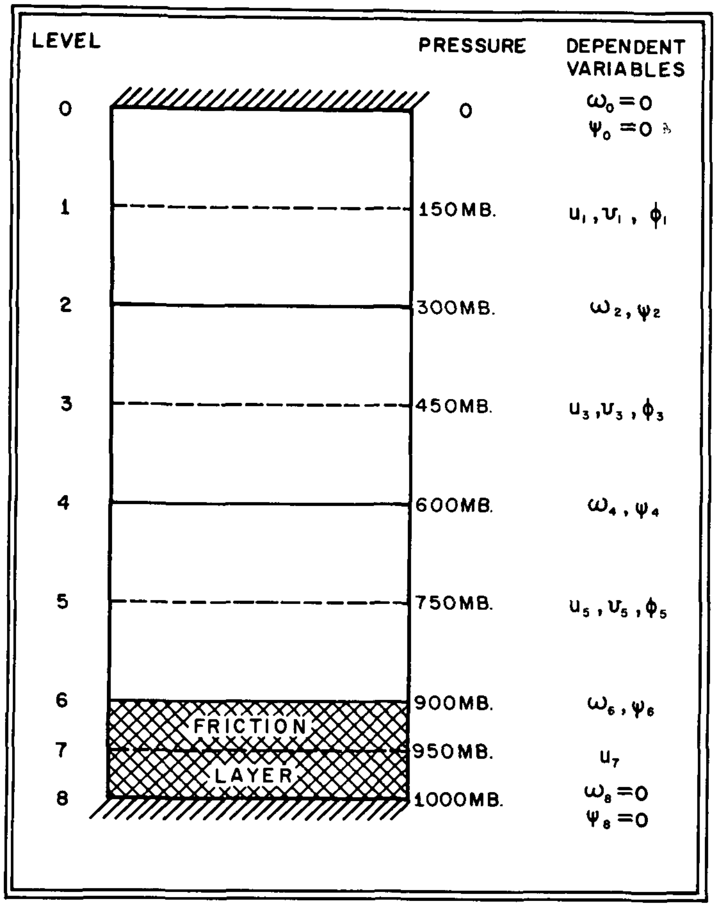

FIGURE 1.—Vertical structure of the model.

$$\frac{\partial \phi_5}{\partial t}-\frac{\partial \phi_3}{\partial t}+\sigma_4 \Delta p\, \omega_4=-\frac{R}{2c_p}\dot{Q}_4 \tag{2}$$

where $\Delta p=300$ mb. We will consider only positive static stability ($\sigma_2>0$, $\sigma_4>0$).

For circular symmetry and balanced motion, the radial equations of motion, after linearization, reduce to the geostrophic wind law,

$$fv_j=\frac{\partial \phi_j}{\partial r},\ j=1, 3, 5 \tag{3}$$

where v_j is the perturbation of the tangential velocity. The centrifugal terms are not present due to their nonlinearity. The equations of tangential motion written for the region above the boundary layer are:

$$\frac{\partial v_j}{\partial t}=-fu_j,\ j=1, 3, 5. \tag{4}$$

f is the Coriolis parameter and u_j is the perturbation of the radial wind.

The Ekman equation for the tangential balance of forces in the friction layer is

$$u\left(f+\frac{v}{r}\right)+g\frac{\partial \tau}{\partial p}=0.$$

τ is the tangential stress. This equation is applied to level 7; the friction term is approximated by

$$g\frac{\partial \tau}{\partial p}=g\tau_8/\delta p$$

and the inertial term is linearized. Thus

$$fu_7+\frac{g\tau_8}{\delta p}=0 \tag{5}$$

where δp (100 mb.) is the thickness of the boundary layer. The stress is approximated by

$$\tau_8=\bar{\rho}_8 C_D\sqrt{v_8^2+u_8^2}\, v_8 \tag{6}$$

where $\bar{\rho}_8$ is the 1000-mb. base-state density and C_D is the drag coefficient. τ_8 is linearized by replacing $\sqrt{v_8^2+u_8^2}$ with V^* (a constant with dimensions of velocity) and, as is common in CISK analyses [1, 9], v_8 is replaced by a fraction of the tangential wind closest to the friction layer. Hence,

$$v_8=lv_5 \tag{7}$$

and, from equation (5),

$$fu_7+Fv_5=0 \tag{8}$$

where

$$F=g\bar{\rho}_8 lC_D V^*/\delta p. \tag{9}$$

Equation (8), at best, is an extremely crude approximation to the boundary layer dynamics. It is, however, typical of those used in models of this type.

The continuity equations for levels 1, 3, 5, and 7 are

$$\omega_{j+1}=\omega_{j-1}-\frac{\Delta p}{r}\frac{\partial}{\partial r}(ru_j),\ j=1, 3, 5 \tag{10a}$$

and

$$\omega_8=\omega_6-\frac{\delta p}{r}\frac{\partial}{\partial r}(ru_7). \tag{10b}$$

From the boundary conditions,

$$\omega_0=\omega_8=0 \tag{11}$$

and the continuity equations, we find

$$\Delta p(u_1+u_3+u_5)+u_7\delta p=0. \tag{12}$$

Equations (10a), (10b), (11), and (12) allow us to introduce a Stokes stream function (ψ) such that

$$\omega_j=\frac{1}{r}\frac{\partial \psi_j}{\partial r},\ j=2, 4, 6 \tag{13}$$

$$u_j=-\frac{(\psi_{j+1}-\psi_{j-1})}{r\Delta p},\ j=1, 3, 5 \tag{14}$$

$$u_7=\frac{\psi_6}{r\delta p} \tag{15}$$

where

$$\psi_0=\psi_8=0.$$

These relationships are used to eliminate reference to the radial and vertical motions in equations (1), (2), (3), (4), and (8). This yields,

$$\frac{\partial v_j}{\partial t}=f\frac{(\psi_{j+1}-\psi_{j-1})}{r\Delta p},\quad j=1,\,3,\,5 \tag{16}$$

$$fv_j=\frac{\partial \phi_j}{\partial r},\quad j=1,\,3,\,5 \tag{17}$$

$$\frac{f\psi_6}{r\Delta p}+Fv_5=0 \tag{18}$$

$$\frac{\partial \phi_3}{\partial t}-\frac{\partial \phi_1}{\partial t}+\frac{\sigma_2\Delta p}{r}\frac{\partial \psi_2}{\partial r}=-\frac{R}{c_p}\dot{Q}_2 \tag{19}$$

$$\frac{\partial \phi_5}{\partial t}-\frac{\partial \phi_3}{\partial t}+\frac{\sigma_4\Delta p}{r}\frac{\partial \psi_4}{\partial r}=-\frac{R}{2c_p}\dot{Q}_4 \tag{20}$$

which is a closed system for V_1, V_3, V_5, ψ_2, ψ_4, ψ_6, ϕ_1, ϕ_3, and ϕ_5 provided that $\dot{Q}_2$ and $\dot{Q}_4$ are expressed in terms of the other dependent variables.

3. THE HEATING FUNCTIONS

The total heating in a vertical column is assumed to be equal to the large-scale upward flux of latent heat through the top of the friction layer. In linear form,

$$\int_0^{p_6}\dot{Q}\frac{dp}{g}=-\frac{L\overline{q}_6\omega_6}{g},\quad \omega_6<0 \tag{21a}$$

$$\dot{Q}=0,\quad \omega_6\geq 0 \tag{21b}$$

where L is the latent heat of condensation and $\overline{q}_6$ is the base-state specific humidity at p_6. These relationships express conditional heating. For unconditional heating,

$$\int_0^{p_6}\dot{Q}\frac{dp}{g}=-\frac{L\overline{q}_6\omega_6}{g} \tag{22}$$

regardless of the sign of ω_6. Heating will be limited to the layer extending from p_1 to p_5. Equations (21a) and (22) may then be approximated by

$$\Delta p(\dot{Q}_2+\dot{Q}_4)=-L\omega_6\overline{q}_6. \tag{23}$$

With the definition

$$\nu\equiv\dot{Q}_2/\dot{Q}_4,\quad \nu\geq 0 \tag{24}$$

and by use of (23), (24), and (13), we obtain

$$\dot{Q}_4=-\frac{L\overline{q}_6}{r\Delta p(\nu+1)}\frac{\partial \psi_6}{\partial r} \tag{25}$$

and

$$\dot{Q}_2=-\frac{\nu L\overline{q}_6}{r\Delta p(\nu+1)}\frac{\partial \psi_6}{\partial r}. \tag{26}$$

The parameter ν, which measures the ratio of upper tropospheric to lower tropospheric heating, is extremely important in the determination of the growth rate and structure of the solutions. For conditional heating, by agreement, we set $L=0$ when $\omega_6\geq 0$.

By use of (16), (17), (18), (19), (20), (25), and (26) and noting that $\Delta p/\delta p=3$, we have

$$\frac{\partial \phi_5}{\partial t}+\frac{\partial \phi_3}{\partial t}+\frac{\partial \phi_1}{\partial t}+\frac{F}{3}\phi_5=0, \tag{27}$$

$$r\left(\frac{\partial \phi_3}{\partial t}-\frac{\partial \phi_1}{\partial t}\right)+S_2\frac{\partial}{\partial r}\left(r\frac{\partial^2 \phi_1}{\partial r\,\partial t}\right)=-\frac{2\Delta p^2\epsilon\overline{q}_6}{3f^2}\left(\frac{\nu}{\nu+1}\right)F\frac{\partial}{\partial r}\left(r\frac{\partial \phi_5}{\partial r}\right), \tag{28}$$

$$r\left(\frac{\partial \phi_5}{\partial t}-\frac{\partial \phi_3}{\partial t}\right)+S_4\frac{\partial}{\partial r}\left\{r\left(\frac{\partial^2 \phi_3}{\partial r\,\partial t}+\frac{\partial^2 \phi_1}{\partial r\,\partial t}\right)\right\}=-\frac{\Delta p^2\epsilon\overline{q}_6}{3f^2(\nu+1)}F\frac{\partial}{\partial r}\left(r\frac{\partial \phi_5}{\partial r}\right), \tag{29}$$

where

$$S_2=\frac{\Delta p^2\sigma_2}{f^2}, \tag{30}$$

$$S_4=\frac{\Delta p^2\sigma_4}{f^2}, \tag{31}$$

and

$$\epsilon=\frac{RL}{2c_p\Delta p^2}. \tag{32}$$

4. SOLUTIONS FOR UNCONDITIONAL HEATING

The solution for unconditional heating will be examined in some detail since it serves as a direct comparison to Syōno and Yamasaki's [9] work and since it provides insight to the more complex solution for conditional heating. For unconditional heating,

$$\phi_j=e^{qt}H_jJ_0(kr),\qquad j=1,3,5 \tag{33}$$

where H_j are the amplitude constants and the eigenvalues of q are given by

$$q=\frac{F}{3}\left\{\frac{\frac{\overline{q}_6\Delta p^2}{f^2}\left[S_2\left(\frac{\epsilon}{\nu+1}-Y_4\right)k^4+(2\epsilon-2Y_4-Y_2)k^2\right]-1}{k^4S_2S_4+2k^2(S_2+S_4)+3}\right\} \tag{34}$$

where

$$Y_4=\frac{f^2S_4}{\overline{q}_6\Delta p^2}=\frac{\sigma_4}{\overline{q}_6}, \tag{35}$$

$$Y_2=\frac{f^2S_2}{\overline{q}_6\Delta p^2}=\frac{\sigma_2}{\overline{q}_6}. \tag{36}$$

$J_0(\eta)$ is the zero order Bessel function of the first kind,

$$k\equiv\lambda/D \tag{37}$$

where $\lambda(\approx 2.4048)$ is the first zero of $J_0(\eta)$ and

$$0<r<D$$

is the domain in which the heating functions are positive.

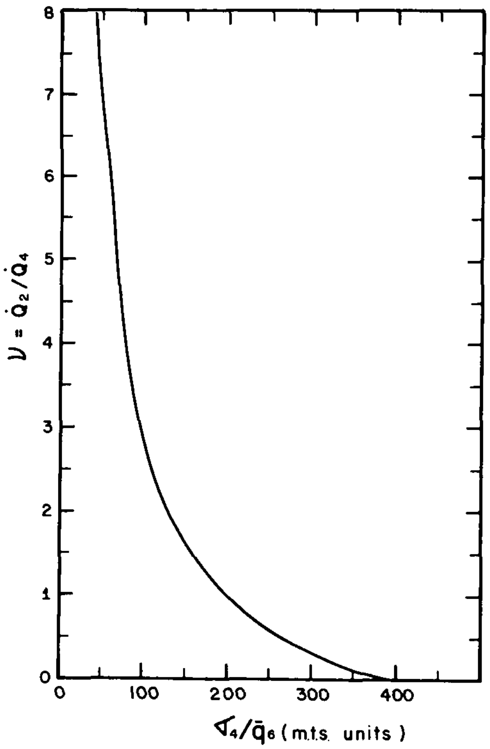

FIGURE 2.—Critical values of ν calculated from (39). ν measures the ratio of upper to lower tropospheric heating. When ν exceeds the critical value, relatively small-scale disturbances are damped.

Equation (34) is similar to (5.7) of Syōno and Yamasaki's [9] paper. The denominator of (34) is positive for $\sigma_2>0$, $\sigma_4>0$. Stability or instability is, therefore, determined by the sign of the numerator. It is clear that disturbances of very large scale (small k) are damped. When

$$0<Y_4<\epsilon \tag{38}$$

and

$$\nu<\frac{\epsilon-Y_4}{Y_4}, \tag{39}$$

the coefficient of k^4 in the numerator of (34) is positive. Under these circumstances, therefore, disturbances of sufficiently small scale will grow and larger scale systems will be damped. This type of spectrum (hereafter, "Type A") is similar to that found from models with only one thermodynamic equation (see [1, 7]). Critical values of ν, calculated from (39), are shown by figure 2.

When ν exceeds this critical value, the coefficient of k^4 in the numerator of (34) is negative and, hence, disturbances of both sufficiently small and sufficiently large scale are damped. When this is the case, growth may occur over an intermediate range of scales provided that the coefficient of k^2 is positive and that the numerator of (34), when considered as a quadratic for k^2, has a nonnegative radical. A spectrum with this shape (decay at very large and very small scales with intermediate growth) will be referred to as Type B. The necessary and sufficient conditions for Type B spectra are:

$$0<Y_4<\epsilon, \tag{40}$$

$$\nu>\frac{\epsilon-Y_4}{Y_4}, \tag{41}$$

which renders the coefficient of k^4 in the numerator of (34) negative,

$$0<Y_2<2(\epsilon-Y_4) \tag{42}$$

which insures that the coefficient of k^2 is positive, and

$$[2(\epsilon-Y_4)-Y_2]^2+4Y_2\left[\frac{\epsilon}{\nu+1}-Y_4\right]\geq 0 \tag{43}$$

which insures that the numerator of (34) will not have a negative radical.

Although not mentioned by Syōno and Yamasaki [9], Type B spectra are implicit in their equation (5.7). However, as will be shown below, growth rates for Type B spectra are very small and this mode of growth appears to have no real importance.

Sample spectra are shown by figure 3.[3] Figure 3a shows results for static stabilities and boundary layer humidity close to Jordan's [4] mean hurricane season sounding. For this case, instability occurs only for $\nu\lesssim 1$ and the spectra are always of Type A. The decrease of growth rate with increasing ν, as illustrated by the figure, was also found by Syōno and Yamasaki [9] and is an extremely important result. We note that the growth rates with ν-values of 0.6 and 0.8 are extremely small (corresponding to e-folding times of about 11 and 28 days, respectively). With ν equal to zero (all heating in the lower troposphere) the e-folding time is about 3 days. One might than conclude realistic solutions could be obtained by use of a ν somewhere between 0.6 and 0.0.

However, as pointed out in section 1, the model must also reproduce the correct structure of tropical cyclones. The observed thermal structure is such that greatest warmth is found in the upper troposphere. Radial temperature gradients in the low troposphere are usually weak and little vertical shear of the tangential wind is observed at low levels.

Although the derivation will not be reproduced here, it may be shown from an examination of the amplitude constants (H_1, H_3, and H_5) that a necessary condition for cyclonic flow to extend into the middle troposphere is

$$\nu>\frac{Y_2}{2Y_4}. \tag{44}$$

When (44) is not satisfied, the solutions give so much warmth in the lower troposphere that the low level

[3] For all numerical results presented: $f=5\times10^{-5}$ sec.$^{-1}$; $C_D=2.5\times10^{-3}$, $l=0.8$, $V^*=5$ m. sec.$^{-1}$

cyclonic flow gives way to anticyclonic flow at 450 mb. For the static stabilities and humidity used to construct figure 3a, (44) is satisfied for values of ν in excess of about 0.83. However, as mentioned above, the growth rates for such values of ν are much too small to be realistic. We must, therefore, conclude that there are no satisfactory values of ν for the mean sounding.

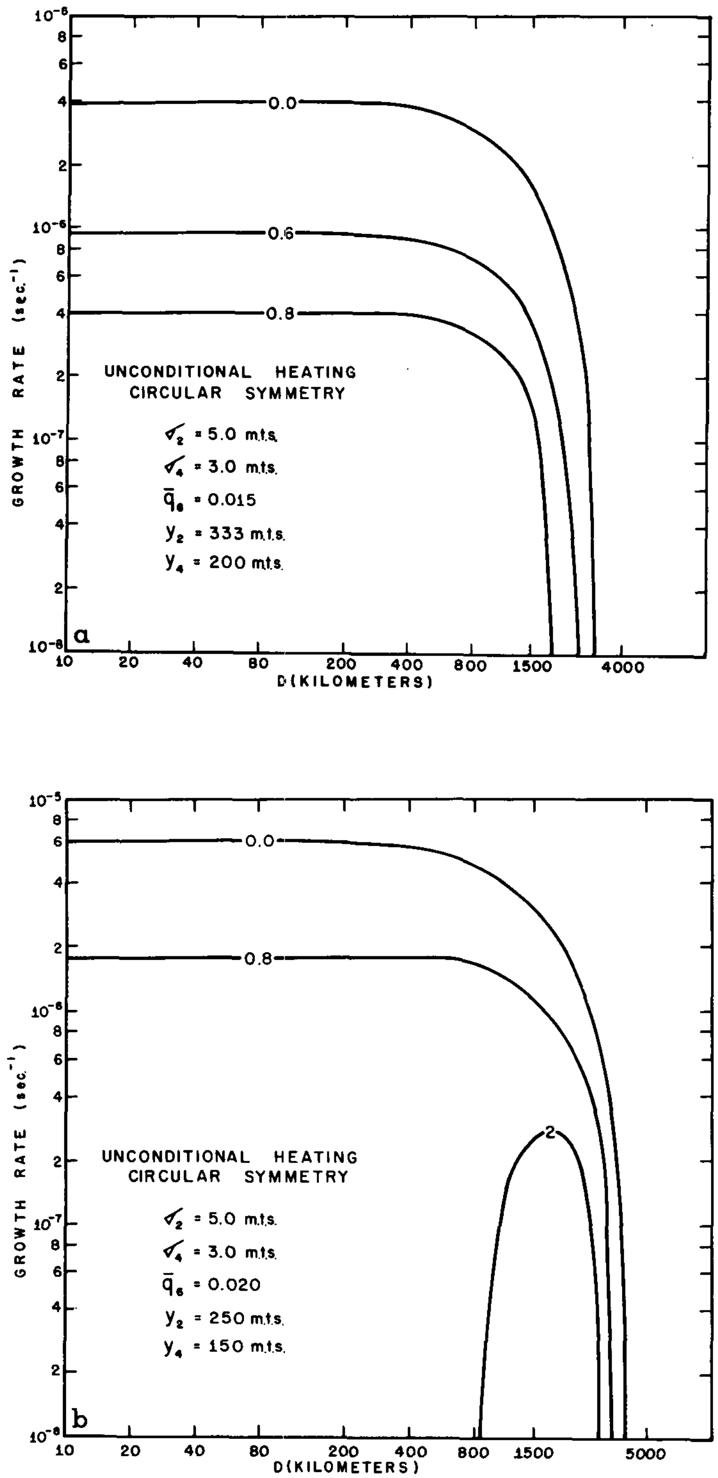

FIGURE 3.—Growth rates for unconditional heating. Curves are labeled with appropriate values of ν. (a) σ_2=5.0 m.t.s., σ_4=3.0 m.t.s., $\bar{q}_6$=.015, (b) σ_2=5.0 m.t.s., σ_4=3.0 m.t.s., $\bar{q}_6$=0.020.

To construct figure 3b, we have used the same static stabilities but the boundary layer humidity is close to its saturation value. With ν=0.8 the e-folding time is about 5 days. As noted in the previous paragraph, ν of about 0.83 gives zero tangential velocity in the middle troposphere. To obtain a more realistic circulation, it turns out that ν must be increased to at least 1. However, growth rates are then too small. There seems to be no reasonable compromise within the framework of the climatological static stabilities. We will have more to say on this matter in the next section which deals with conditional heating.

For completeness, figure 3b shows the spectrum with ν=2. This is a Type B spectrum. The e-folding time for the maximum growth rate is about 37 days, thus verifying the statement made earlier concerning the insignificance of this mode of growth.

5. SOLUTIONS FOR CONDITIONAL HEATING

Equations (25) and (26) are now valid only for $0<r<D$. At greater radii, $\dot{Q}_2=\dot{Q}_4=0$. Separate solutions are obtained for the heated and nonheated regions. At $r=D$, these solutions must mesh through the dynamic and kinematic boundary conditions.

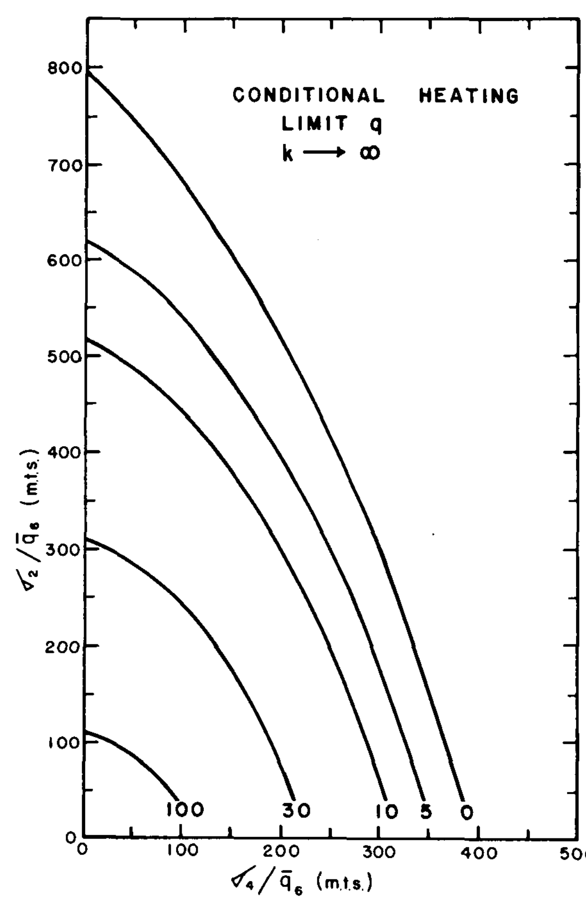

FIGURE 4.—Growth rates (conditional heating) in the limiting case as the radius of the heated region approaches zero. Isopleths are labeled in units of 10^{-7} sec.$^{-1}$

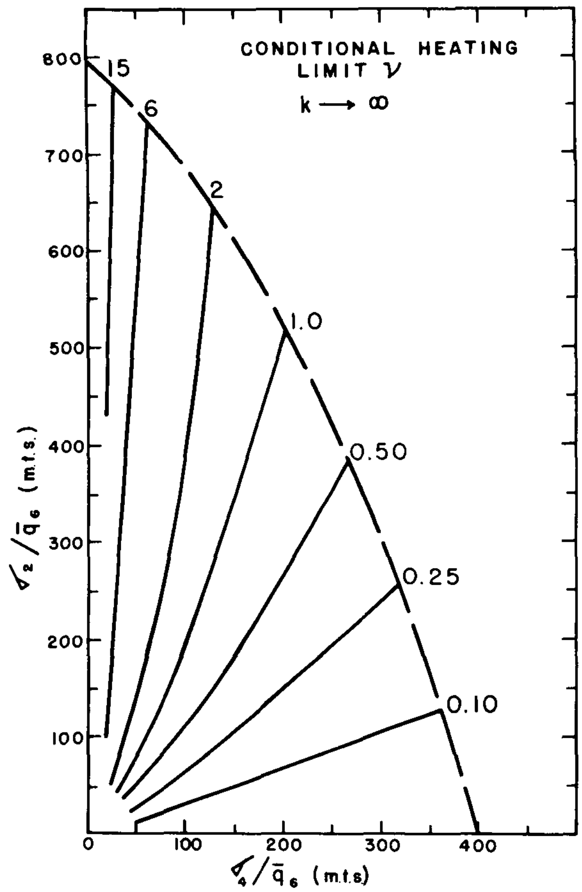

FIGURE 5.—Eigenvalues for ν in the limiting case as the radius of the heated region approaches zero.

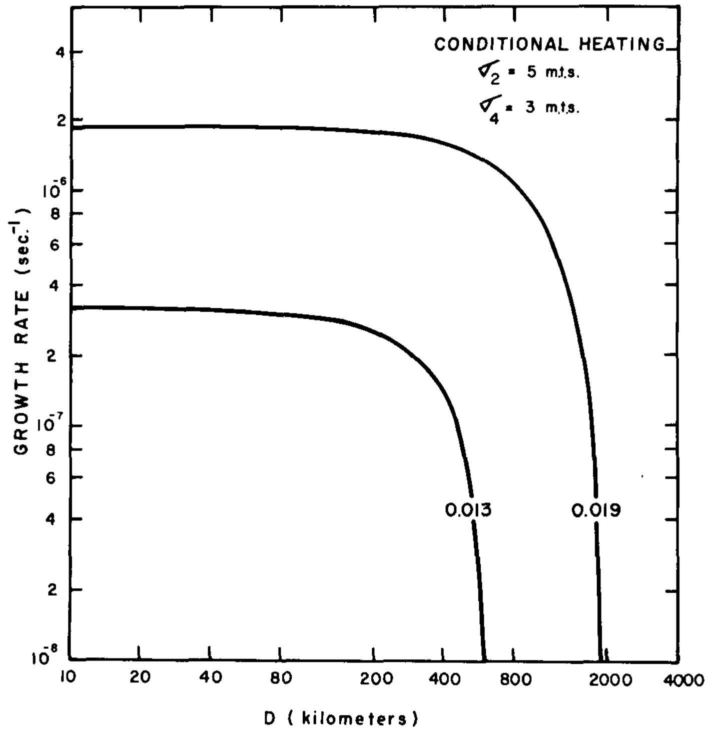

FIGURE 6.—Sample growth rate curves for conditional heating $\sigma_2=5.0$ m.t.s., $\sigma_4=3.0$ m.t.s. Curves are labeled with the appropriate values of $\bar{q}_6$.

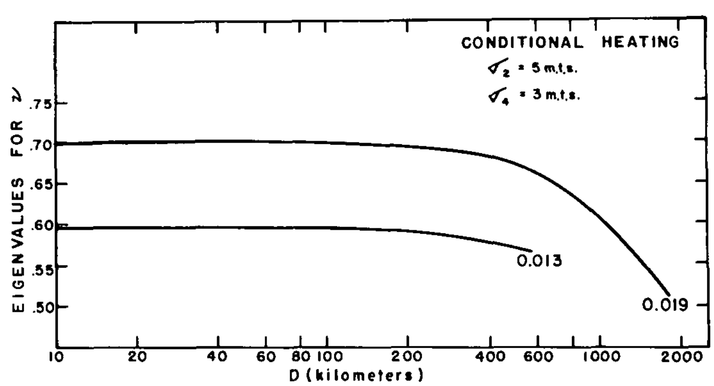

FIGURE 7.—Eigenvalues of ν as a function of D; $\sigma_2=5.0$ m.t.s., $\sigma_4=3.0$ m.t.s. Curves are labeled with the appropriate values of $\bar{q}_6$.

In the heated region, (33) gives the dependent variables provided that (34) holds. (Equation (37) *does not* hold for conditional heating.) For $r>D$, we suppress the heating terms in (28) and (29) and proceed as before. The solutions which permit growth and remain finite as r becomes large are:

$$\phi_j=e^{qt}\hat{H}_jK_0(mr), \qquad j=1,3,5 \tag{45}$$

provided that

$$q=\frac{F}{3}\left\{\frac{(2Y_4+Y_2)(S_2m^2)-Y_4(S_2m^2)^2-Y_2}{Y_4(S_2m^2)^2-2(Y_4+Y_2)(S_2m^2)+3Y_2}\right\}. \tag{46}$$

$K_0(\eta)$ is the zero order modified Bessel function of the second kind ([3], p. 155). At $r=D$, continuity of the level 5 geopotential and radial wind require

$$\frac{J_1(kD)}{J_0(kD)}=\frac{m}{k}\frac{K_1(mD)}{K_0(mD)} \tag{47}$$

which is similar to the relationship found in models with less vertical resolution [1, 7]. However, the further requirement that u_1, u_3, u_7, ϕ_1 and ϕ_3 also be continuous at $r=D$ leads to the added restriction

$$q=\frac{F}{3}\left\{\frac{2\epsilon\nu(2-m^2S_2)}{Y_2(m^2+k^2)(\nu+1)}-1\right\}. \tag{48}$$

If ν and D are both specified, the problem is overdetermined since (34), (46), (47), and (48) must all be satisfied; but the only parameters at our disposal are m, k, and q. We are, therefore, not free to specify both ν and D. Once D has been set, ν is an eigenvalue of the problem and (34), (46), (47), and (48) must be solved simultaneously for k, m, ν, and q.

This restriction on ν is surprising and appears to have more mathematical than physical significance. It appears to stem entirely from the requirement that the time dependency of the solutions be the same in the heated and nonheated regions. Unfortunately, closed solutions without this restraint appear to be out of reach. Our

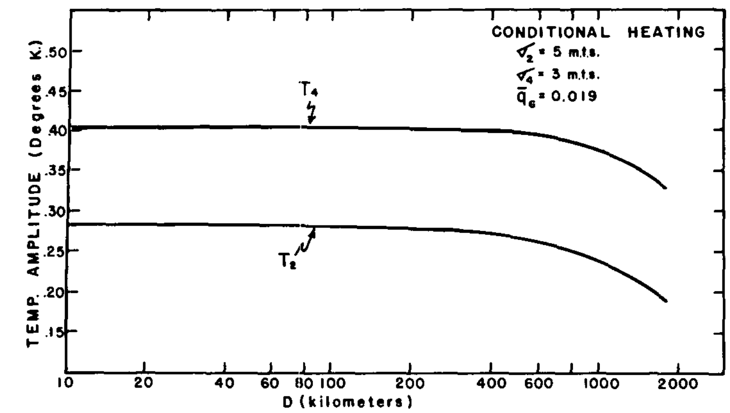

FIGURE 8.—Temperature amplitudes in the heated region as a function of D; $\sigma_2=5.0$ m.t.s., $\sigma_4=3.0$ m.t.s., $\bar{q}_6=0.019$ (H_1 arbitrarily set to 100 m.[2] sec.$^{-2}$).

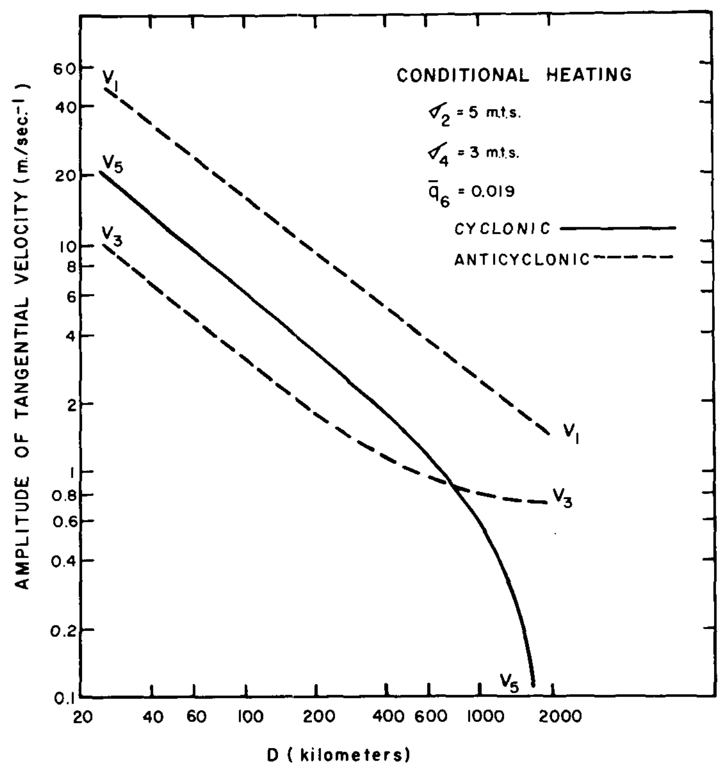

FIGURE 9.—Tangential velocity amplitudes in the heated region as a function of D; $\sigma_2=5.0$ m.t.s., $\sigma_4=3.0$ m.t.s., $\bar{q}_6=0.019$ (H_1 arbitrarily set to 100 m.[2] sec.$^{-2}$).

results from numerical experiments with the nonlinear equations (to be reported on at a later date) indicate that growth may take place for values of ν other than the eigenvalue provided that the critical value given by (39) is not exceeded. Despite this limitation, the solution for conditional heating does allow us to delineate the values of ν and static stability needed for the development of systems resembling tropical cyclones in a fairly direct manner.

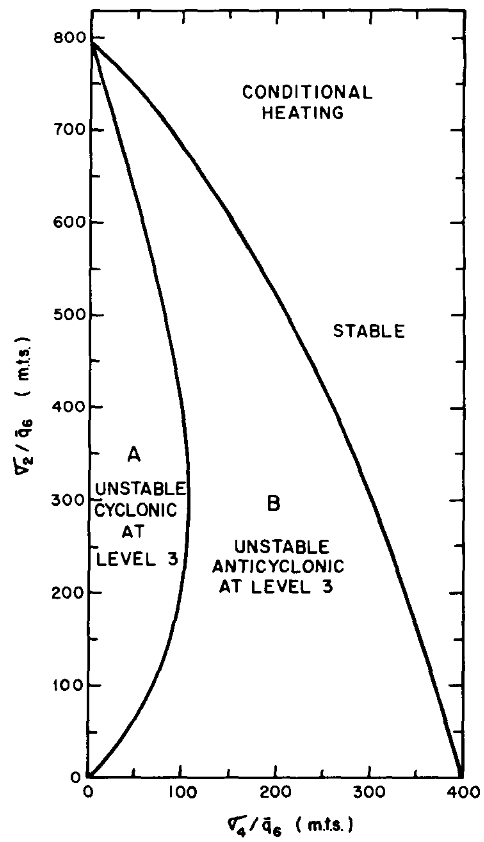

FIGURE 10.—Instability is limited to regions A and B. In the limiting case as D approaches zero, the midtropospheric tangential velocity is anticyclonic in region A and cyclonic in region B.

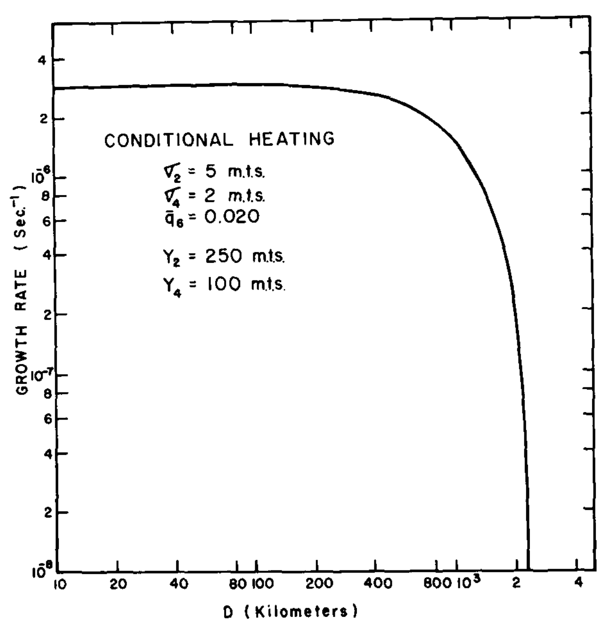

FIGURE 11.—Growth rate as a function of D; $\sigma_2=5.0$ m.t.s., $\sigma_4=2.0$ m.t.s., $\bar{q}_6=0.020$.

By use of (34), (46), and (48) it may be shown that

$$Y_2 < \frac{4\epsilon(\epsilon - Y_4)}{2\epsilon - Y_4} \tag{49}$$

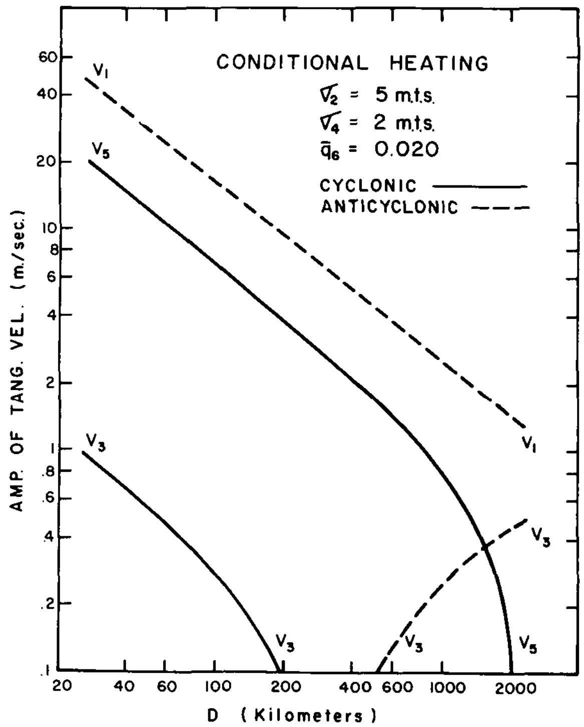

FIGURE 12.—Tangential velocity amplitudes in the heated region as a function of D; $\sigma_2=5.0$ m.t.s., $\sigma_4=2.0$ m.t.s., $\bar{q}_6=0.020$ (H_1 arbitrarily set to 100 m.^{2} sec.$^{-2}$).

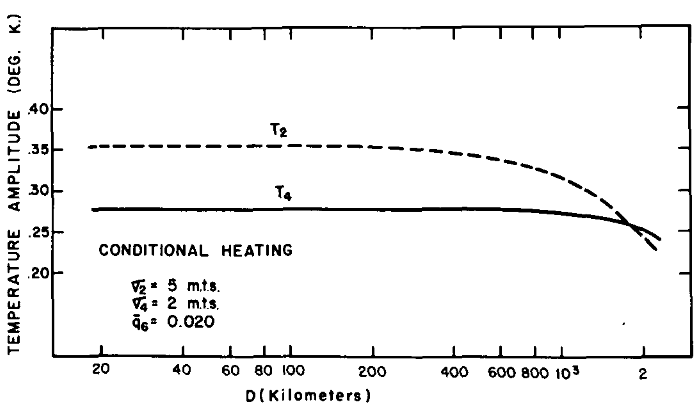

FIGURE 13.—Temperature amplitudes in the heated region as a function of D; $\sigma_2=5.0$ m.t.s., $\sigma_4=2.0$ m.t.s., $\bar{q}_6=0.020$ (H_1 arbitrarily set to 100 m.^{2} sec.$^{-2}$).

is a necessary condition for growth. The algebraic manipulations required to derive (49) are rather tedious and will not be reproduced here.

It may further be shown that only Type A spectra occur for conditional heating. This is verified by figures 4 and 5 which show growth rates and eigenvalues for ν in the limiting cases as D approaches zero.[4] These are obtained by simultaneous solution of (34), (46), and (48) in the limit as k becomes indefinitely large. The eigen-

[4] In this case, k approaches infinity but the product kD approaches zero.

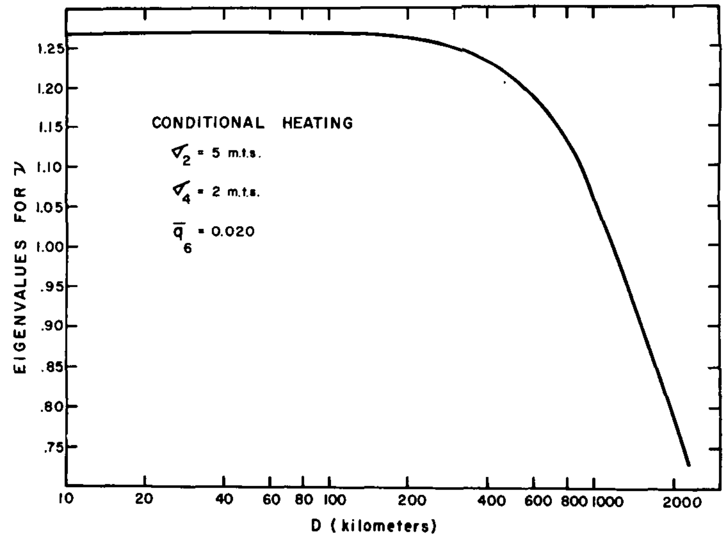

FIGURE 14.—Eigenvalues of ν as a function of D; $\sigma_2=5.0$ m.t.s., $\sigma_4=2.0$ m.t.s., $\bar{q}_6=0.020$.

values for ν (fig. 5) are, of course, less than the critical values given by (39).

Figure 6 shows sample growth rate curves. The static stabilities are appropriate to Jordan's [4] mean sounding. The $\bar{q}_6=0.013$ curve represents mean humidity conditions; $\bar{q}_6=0.019$ represents saturated conditions. In the former case, growth is very slow and the e-folding time is about 37 days. With a saturated boundary layer, growth proceeds more rapidly and the e-folding time is about 6 days which is fairly realistic.

Figure 7 shows the eigenvalues of ν while figures 8 and 9 provide information concerning the structure of the systems ($\bar{q}_6=0.019$ case). The temperature amplitudes are obtained hydrostatically from the equations

$$T_2=(H_1-H_3)/R \tag{50}$$

$$T_4=2(H_3-H_5)/R \tag{51}$$

where R is the gas constant for air. Neither the thermal structure (fig. 8) nor the wind field (fig. 9) resembles that of tropical cyclones. The eigenvalues for ν (fig. 7) fall well below the values needed to obtain a realistic thermal structure (as given by (44)).

Further analysis shows that eigenvalues of ν necessary for realistic structure are limited to Domain A of figure 10 and occur only for smaller values of D. If we retain $\bar{q}_6=0.019$, $\sigma_2=5$ m.t.s. but use $\sigma_4=2$ m.t.s. (33-percent reduction from the mean) we obtain the solution shown by figures 11, 12, 13, and 14. With the exception of the overly vigorous upper tropospheric anticyclonic motion, structure and growth rate are fairly realistic. In real tropical cyclones, nonlinear vertical transport of cyclonic angular momentum appears to play a significant role in the control of the upper tropospheric circulation. Linear models will then always tend to overestimate the anticyclonic character of the upper troposphere.

7. SUMMARY AND CONCLUSIONS

When the thermodynamic equation is applied at two levels, eigenvalue analysis of the CISK mechanism leads to somewhat different conclusions depending upon whether conditional or unconditional heating is treated. For unconditional heating, the ratio of upper to lower tropospheric heating (ν) is arbitrary. Its value plays a crucial role in determining: 1) whether or not disturbances can grow, 2) the shape of the growth rate spectrum, and 3) the structure of growing disturbances.

For given conditions of static stability and humidity, the smaller values of ν yield spectra (similar to that found in models with only one thermodynamic level) in which all disturbances smaller than a critical scale are unstable (Type A). Under certain conditions of humidity and static stability, a second spectral form may be found for larger values of ν. In this case (Type B), growth occurs in an intermediate range of scale; the smallest and largest scales are damped. The Type B spectra have rather small growth rates and appear to be of no great significance. If the static stabilities are those of the mean hurricane season sounding, it is not possible to find a value of ν which provides a reasonable compromise between growth rate and structure. The larger values of ν yield realistic thermal structures but the growth rates are too small. Smaller values of ν give realistic growth rates but a structure in which warmth is concentrated in the lower troposphere to the extent that the low level cyclonic flow decays rapidly with height and anticyclonic flow is present in the midtroposphere.

A surprising aspect of the solution for conditional heating is the result that ν is an eigenvalue and not arbitrary. This appears to have more mathematical than physical significance since initial value formulations may yield growth with values of ν other than the eigenvalue (provided that ν does not exceed the critical value for Type-A growth found in the solution for unconditional heating).

For conditional heating, only Type-A spectra are found. The solutions show that Jordan's mean hurricane season sounding can support only very slow growth; *e*-folding times are on the order of 37 days. The same thermal structure but with the boundary layer humidity increased to saturation gives *e*-folding times of 5 to 6 days. However, the resulting disturbances show thermal structures which are dissimilar to those of observed tropical cyclones. Further analysis shows that the solution is capable of yielding more realistic disturbances if the lower tropospheric static stability is about ⅓ less than that of the mean sounding. Under these conditions, reasonable growth rates and thermal structures are obtained provided that heat is distributed in the vertical such that the ratio of upper to lower tropospheric heating is about 1.25. It would, therefore, seem that theories of cumulus-cyclone interaction must provide a similar vertical distribution of heating if the life cycle of the tropical cyclone is to be realistically simulated.

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